Meshfree volume-averaged nodal pressure methods for incompressible elasticity

Jack S. Hale
Research Unit in Engineering Science, University of Luxembourg, Luxembourg

Christian J. Cyron,
Department of Mechanical Engineering, Yale University, USA

Alejandro Ortiz
Department of Mechanical Engineering, Universidad de Chile, Chile

ACME 2014, Exeter University

RUES | RESEARCH UNIT IN ENGINEERING SCIENCES

The problem of volumetric locking

Find $u_h \in \mathcal{U}_h$ such that

$$\int_\Omega \mathbf{C}\epsilon(\mathbf{u}_h) : \epsilon(\mathbf{v})\ d\Omega = \int_\Gamma \mathbf{f} \cdot \mathbf{v}\ d\Omega \quad \forall v \in H_0^1(\Omega)$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

Why?

$$||u - u_h||_1 \leq C(\lambda) h^p ||u||_{p+1}$$

Options to beat locking:

- Increase polynomial order p
- Decrease mesh size h
- Remove the dependence of constant on λ

Move to a mixed formulation

Find $\mathbf{u}_h \in \mathcal{P}_h$ and $p_h \in \mathcal{P}_h$ such that:

$$\mu \int_\Omega \epsilon(\mathbf{u}_h) \cdot \epsilon(\mathbf{v}) \; d\Omega + \int_\Omega p_h \nabla \cdot \mathbf{v} \; d\Omega = \int_\Omega \mathbf{f} \cdot \mathbf{v} \; d\Omega \qquad \forall \mathbf{v} \in \mathcal{U}_h$$

$$\int_\Omega \nabla \cdot \mathbf{u}_h \; q \; d\Omega - \frac{1}{\lambda} \int_\Omega p_h q \; d\Omega = 0 \qquad \forall q \in \mathcal{P}_h$$

Stability

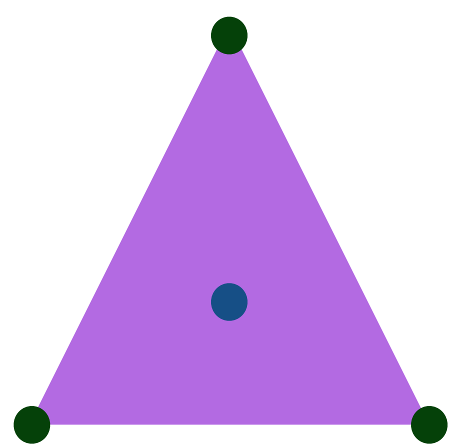
Fix one problem just to introduce another…

$$\inf_{q \in \mathcal{P}_h} \sup_{v \in \mathcal{U}_h} \frac{\int_\Omega q \nabla \cdot \mathbf{v} d\Omega}{||v||_1 ||q||_{0/\mathbb{R}}} \geq \beta_h > 0$$

Key to satisfying this condition is a good ‘balance’ between the pair of spaces…

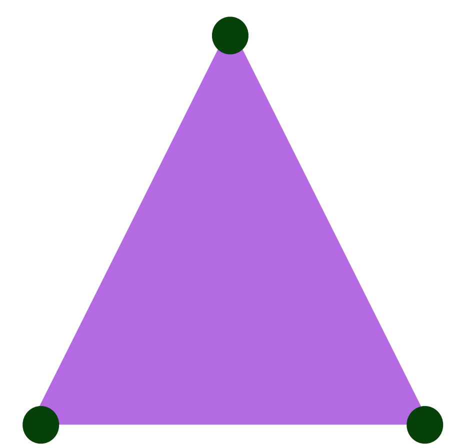
MINI element

$[CG_1 \oplus B_3]^2$



$\mathcal{U}_h$

CG_1

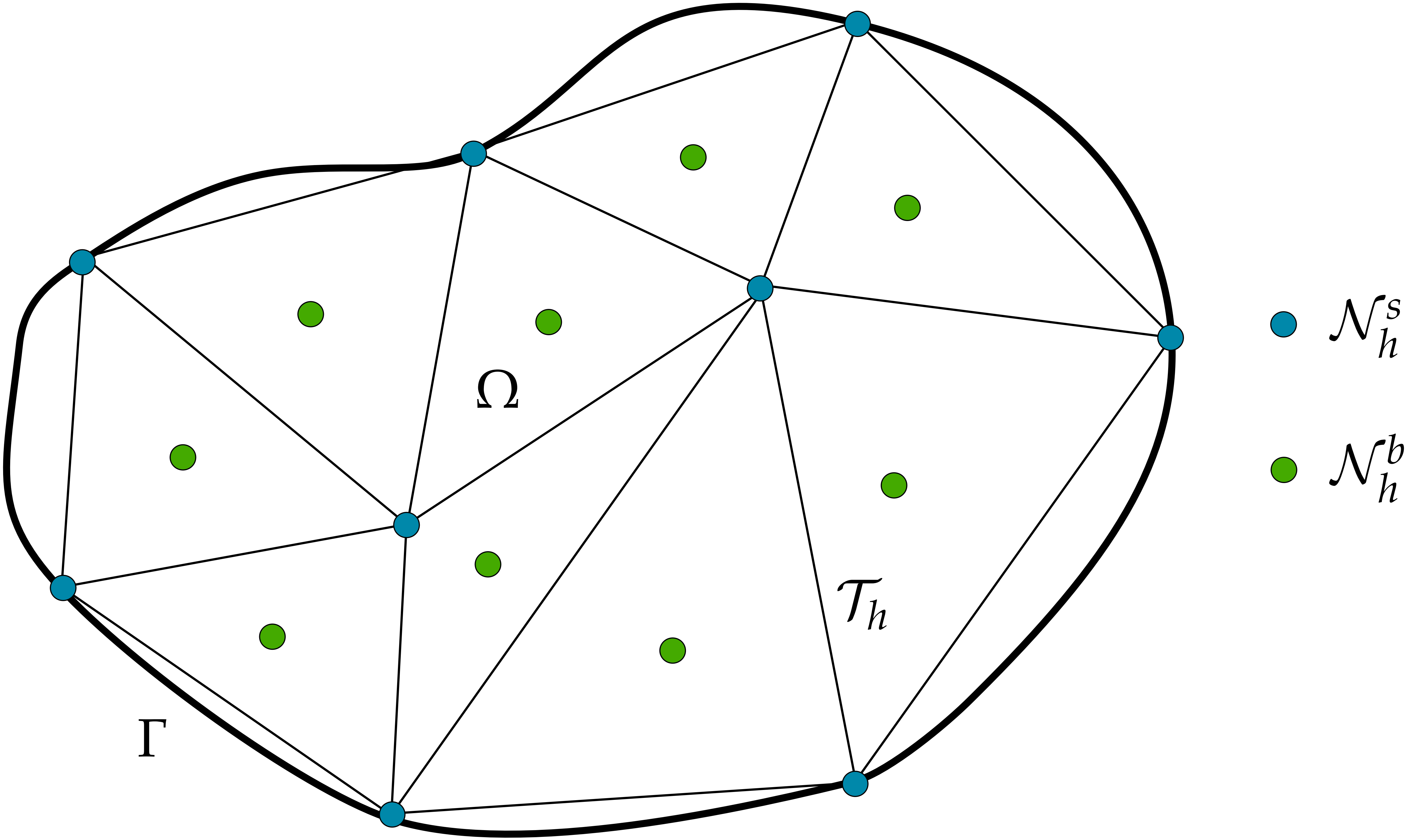


$\mathcal{P}_h$

Arnold, Brezzi, Fortin 1984

Questions

- Can we produce a stable pair of spaces for the mixed formulation using *meshfree* approximation schemes?
- Can we produce a *general* scheme, which works for arbitrary spaces of meshfree basis functions and even finite element basis functions?
- Can we *eliminate* the pressure space to produce a generalised displacement method?
- How does enrichment, in the manner of the MINI bubble, affect the convergence and stability?
- Can we produce a numerical method that is particularly robust with respect to mesh distortion, and therefore ideally suited to hyperelastic problems?



Ω
Γ
$\mathcal{T}_h$
$\mathcal{N}_h^s$
$\mathcal{N}_h^b$

Spaces

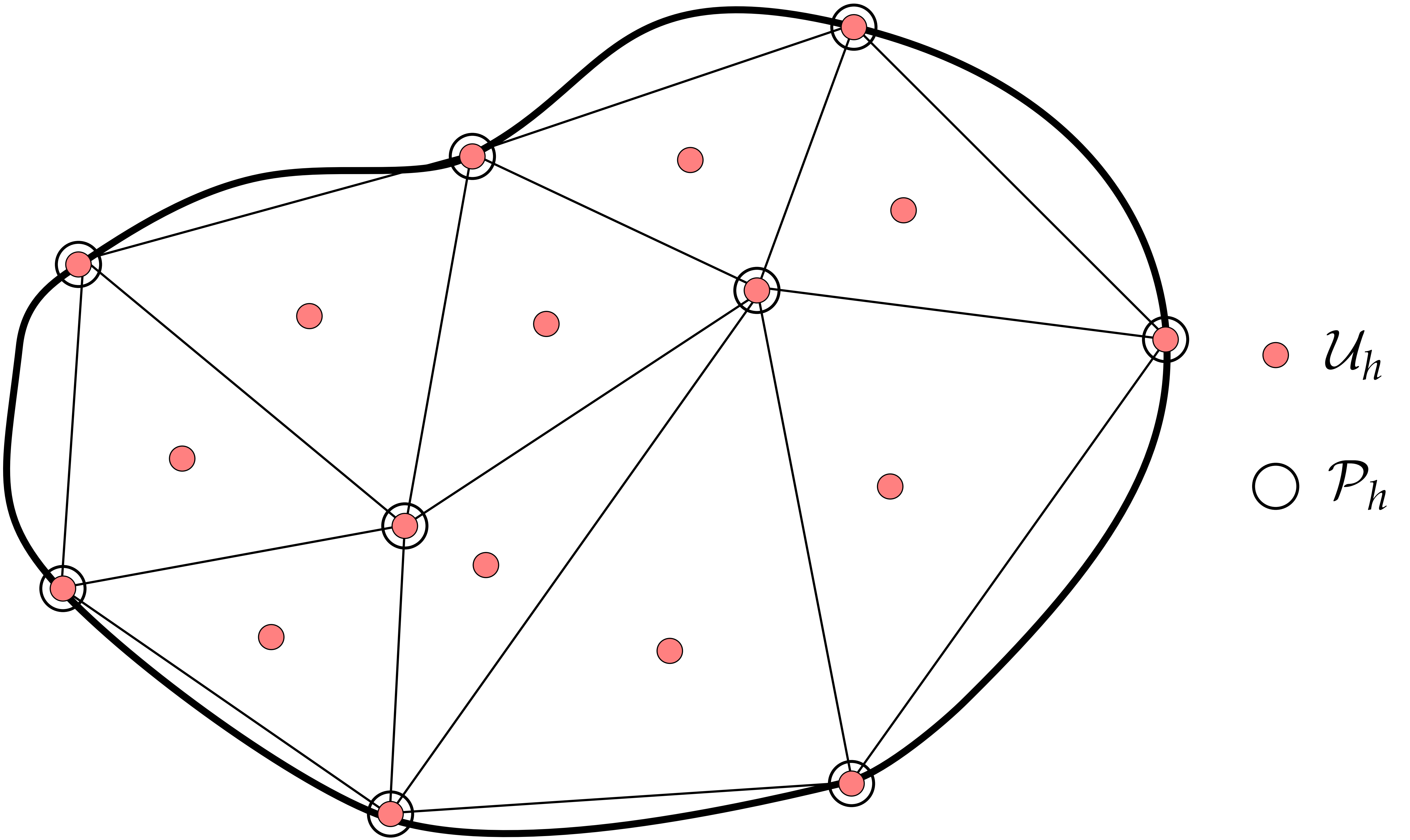
$$\mathcal{U}_h := [ME(\Omega; \mathcal{N}_h, \rho)]^2$$

$$\boldsymbol{u}_h(\boldsymbol{x}) = \sum_{i=1}^{N} \boldsymbol{\phi_i} \boldsymbol{u}_i$$

$$\mathcal{P}_h := CG_1(\Omega; \mathcal{T}_h)$$

For simplicity of the exposition, not required

$$p_h = \sum_{i=1}^{M} N_i p_i$$


$\mathcal{U}_h$
$\mathcal{P}_h$

Saddle-point problem

$$\int_\Omega \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} \, d\Omega \, \boldsymbol{u} + \int_\Omega \boldsymbol{B}^T \mathbf{m} \boldsymbol{N}_p \, d\Omega \, \boldsymbol{p} = \int_\Omega \boldsymbol{\Phi}_{\boldsymbol{u}}^T \boldsymbol{f} \, d\Omega$$

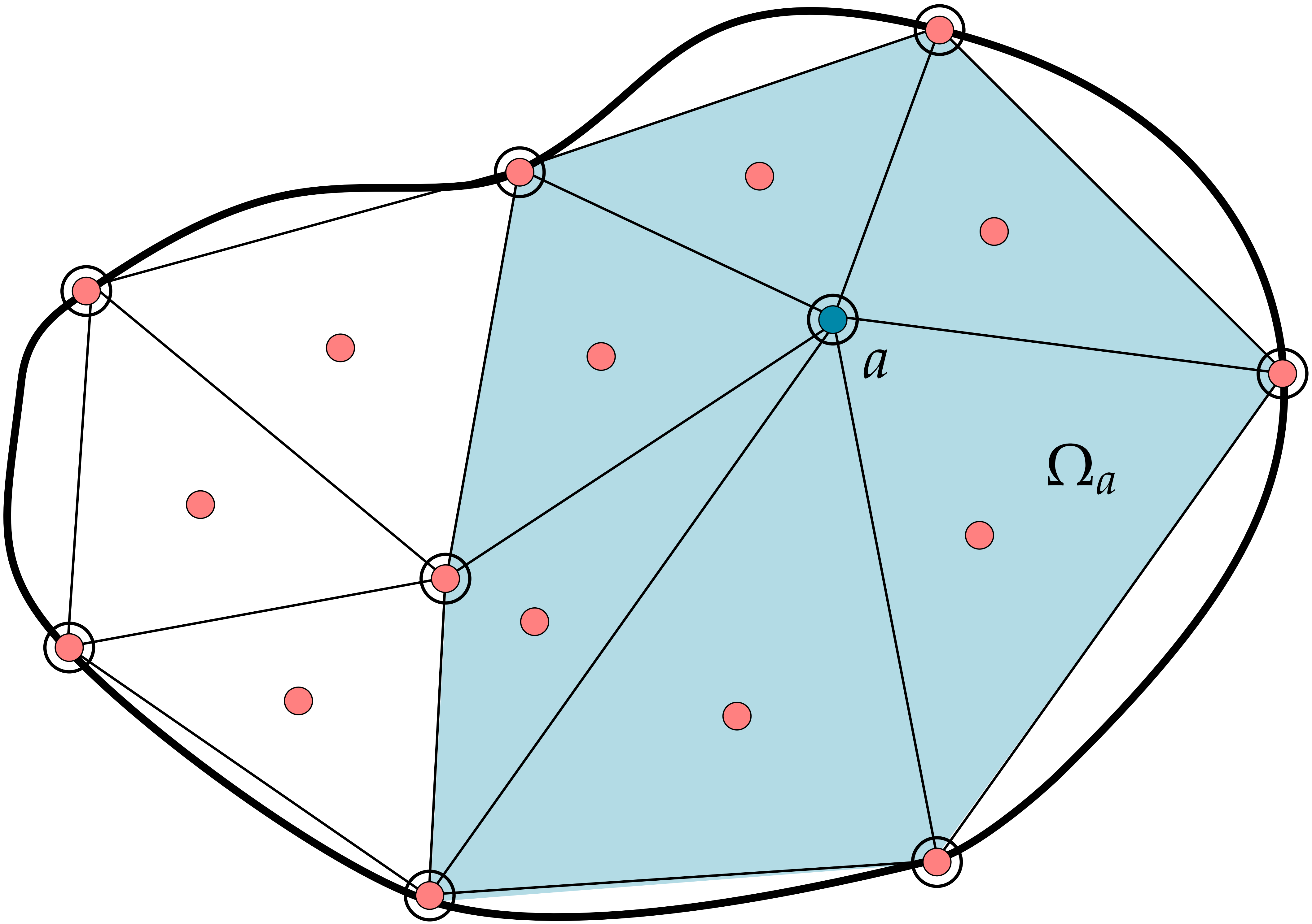
$$\int_\Omega \boldsymbol{N}_p^T \mathbf{m}^T \boldsymbol{B} \, d\Omega \, \boldsymbol{u} - \frac{1}{\lambda_e} \int_\Omega \boldsymbol{N}_p^T \boldsymbol{N}_p \, d\Omega \, \boldsymbol{p} = 0$$

$$\mathbf{B} = \begin{bmatrix} \frac{\partial \phi_{u_x}}{\partial x_1} & 0 \\ 0 & \frac{\partial \phi_{u_y}}{\partial x_2} \\ \frac{\partial \phi_{u_x}}{\partial x_2} & \frac{\partial \phi_{u_y}}{\partial x_1} \end{bmatrix} \quad \mathbf{m} = \begin{Bmatrix} 1 & 1 & 0 \end{Bmatrix}^T \quad \mathbf{D} = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

Saddle-point problem

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & -\mathbf{C} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \mathbf{p} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f} \\ \mathbf{0} \end{Bmatrix}$$

How can we get rid of p?


a
Ω_a

For every pressure node:

$$\sum_{b=1}^{N} \int_{\Omega} \mathbf{N}_{pa} \mathbf{m}^T \mathbf{B}_b \, d\Omega \; \mathbf{u}_b - \frac{1}{\lambda} \sum_{a=1}^{M} \int_{\Omega} \mathbf{N}_{pa} \, d\Omega \; p_a = 0$$

Restrict integration domain to local domain:

$$\sum_{b=1}^{N} \int_{\Omega_a} N_{pa} \mathbf{m}^T \mathbf{B}_b \, d\Omega \; \mathbf{u}_b - \frac{1}{\lambda} \sum_{a=1}^{M} \int_{\Omega_a} N_{pa} \, d\Omega \; p_a = 0$$

Re-arrange to get equation for every pressure dof:

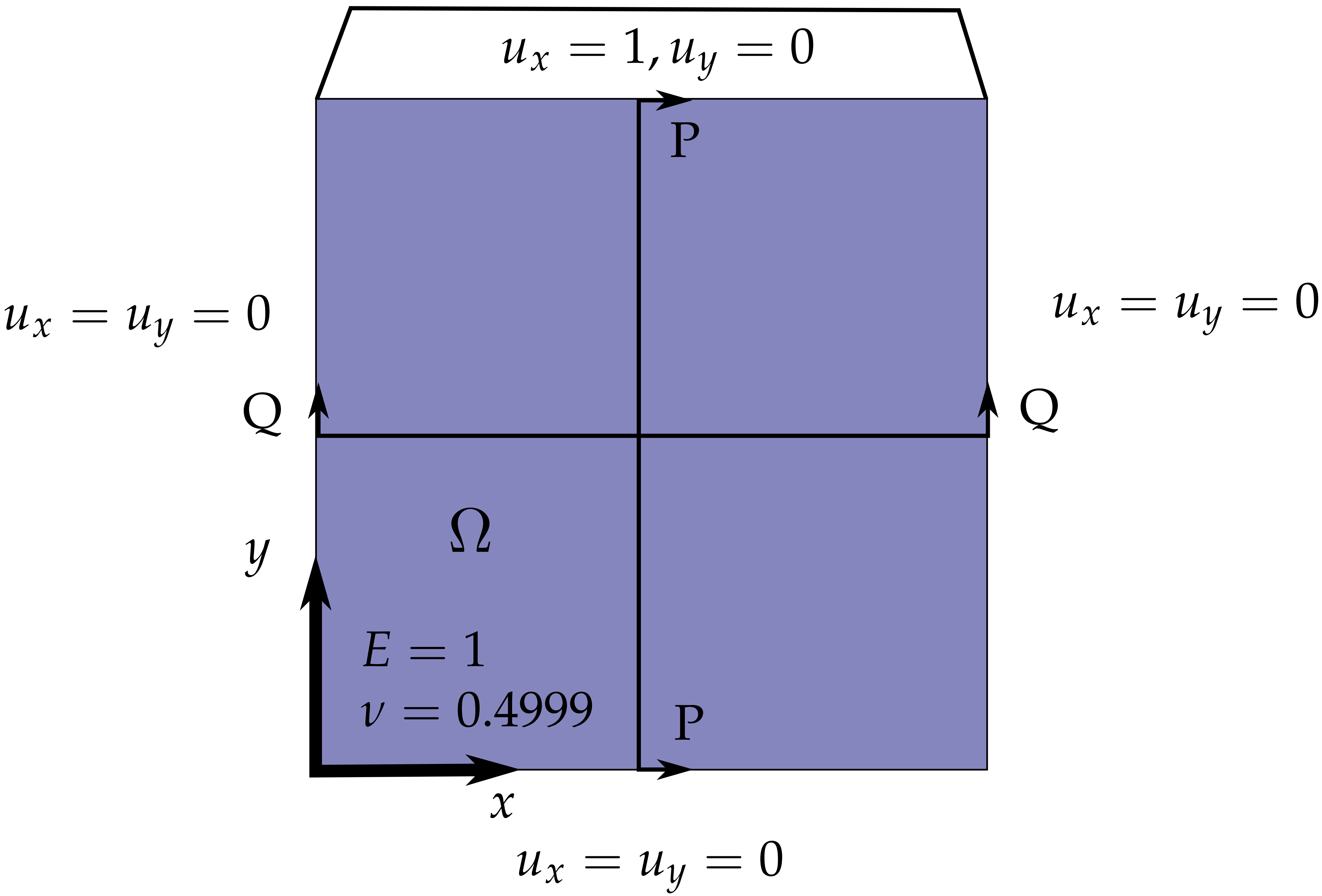
$$p_a = -\lambda \sum_{b=1}^{N} \left\{ \frac{\int_{\Omega_a} N_{pa} \mathbf{m}^T \mathbf{B}_b \, d\Omega}{\int_{\Omega_a} N_{pa} \, d\Omega} \right\} \mathbf{u}_b$$

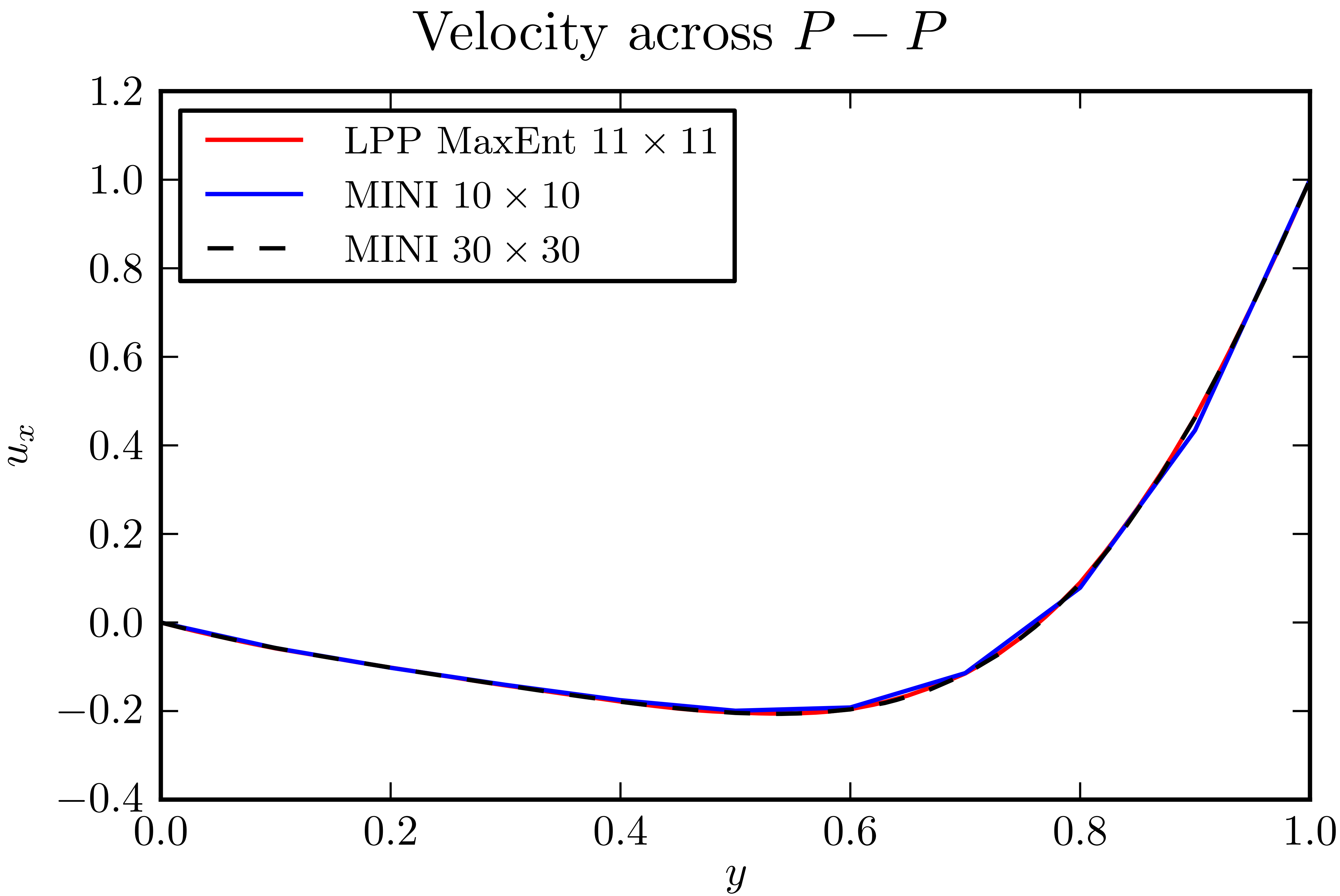
Resulting in…

$$\mathbf{p} = \mathbf{Q}\mathbf{u}$$

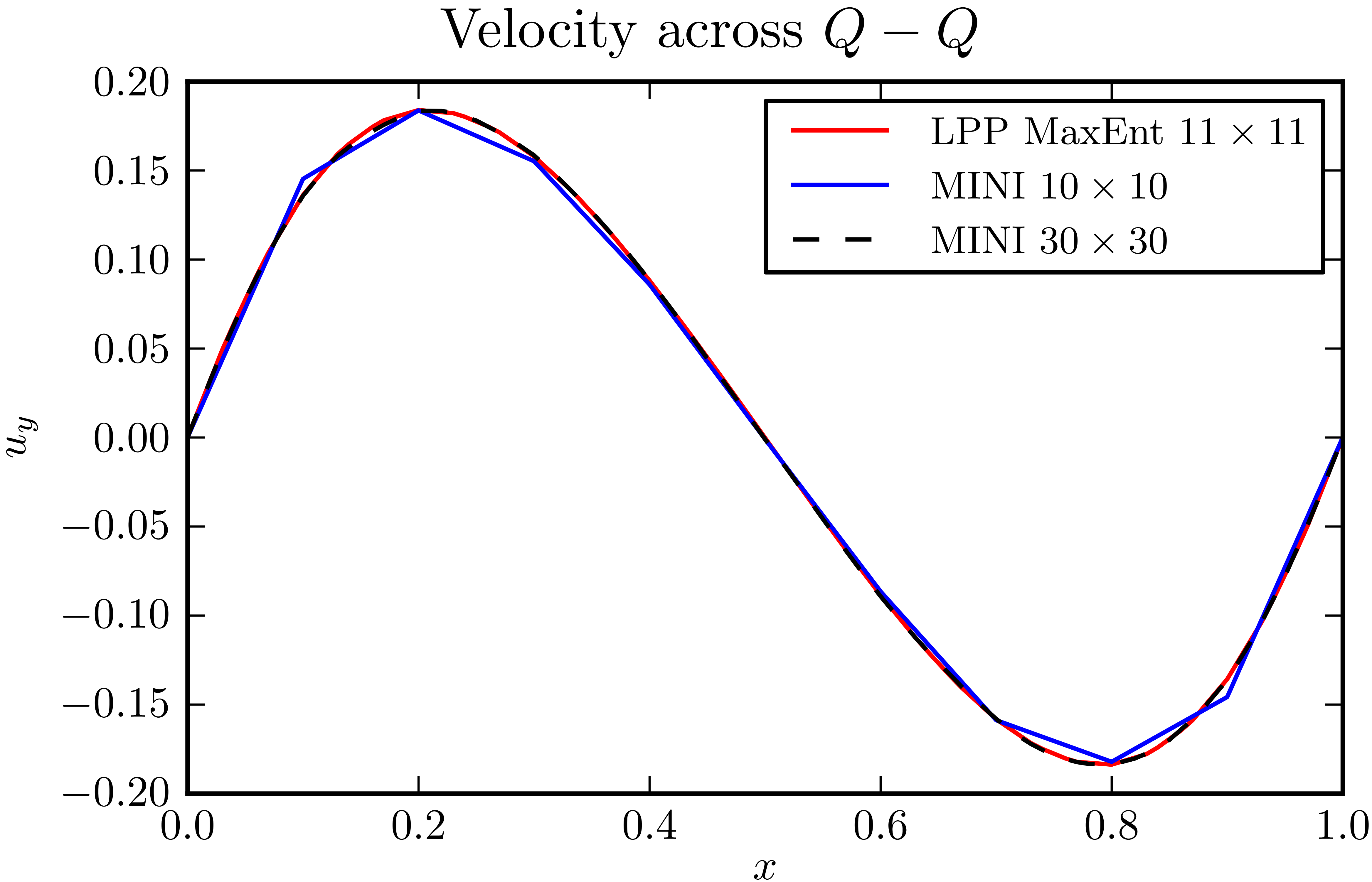
$$\int_{\Omega} \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} \; d\Omega \; \boldsymbol{u} + \int_{\Omega} \boldsymbol{B}^T \mathbf{m} \boldsymbol{N}_p \mathbf{Q} \; d\Omega \; \boldsymbol{u} = \int_{\Omega} \boldsymbol{\Phi}_{\boldsymbol{u}}^T \boldsymbol{f} \; d\Omega$$

Leaky-lid cavity flow



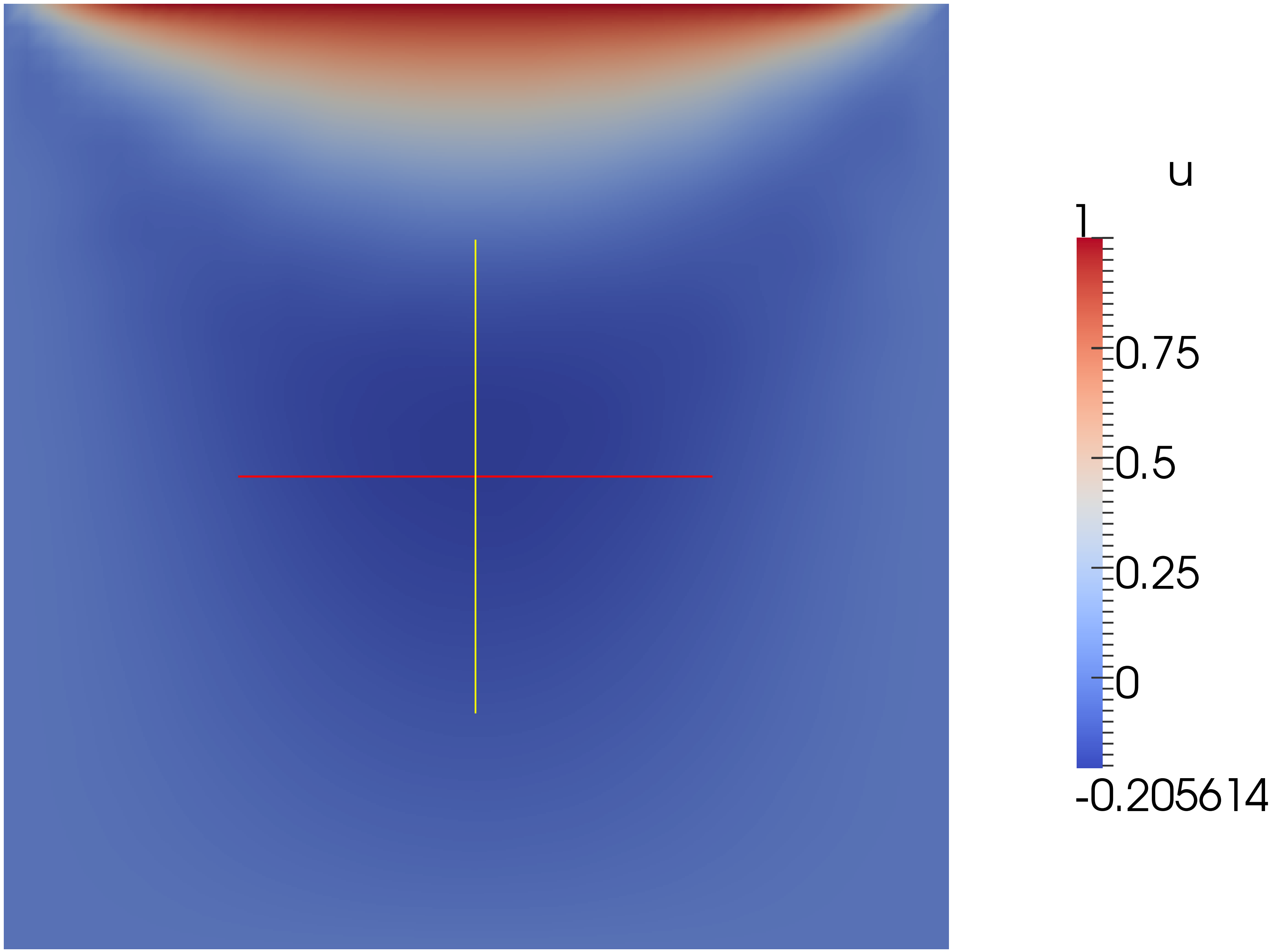


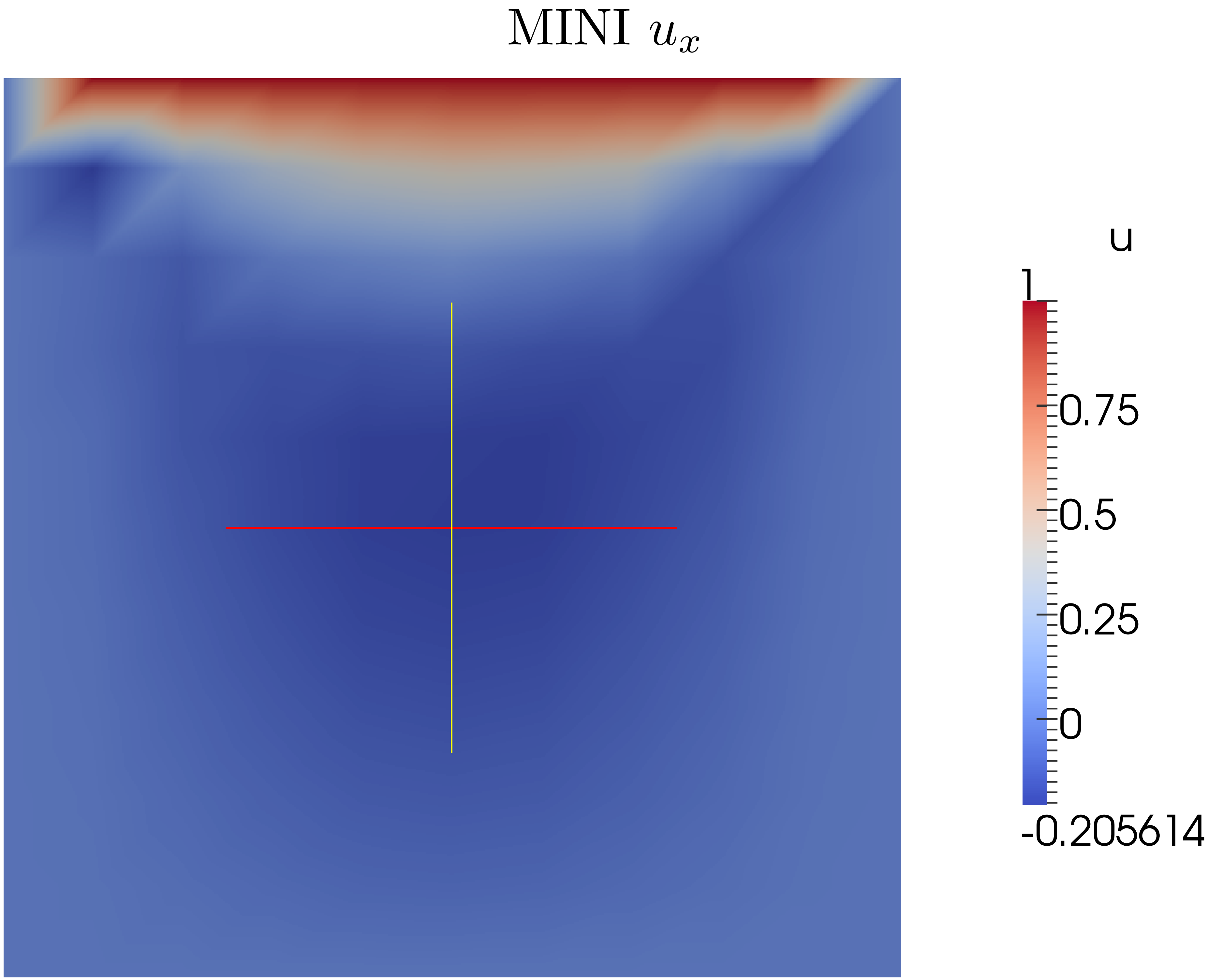
Velocity across $P - P$
LPP MaxEnt 11×11
MINI 10×10
MINI 30×30
u_x
y
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
0.0
0.2
0.4
0.6
0.8
1.0



Velocity across $Q-Q$
LPP MaxEnt 11×11
MINI 10×10
MINI 30×30
0.20
0.15
0.10
0.05
0.00
−0.05
−0.10
−0.15
−0.20
u_y
0.0
0.2
0.4
0.6
0.8
1.0
x

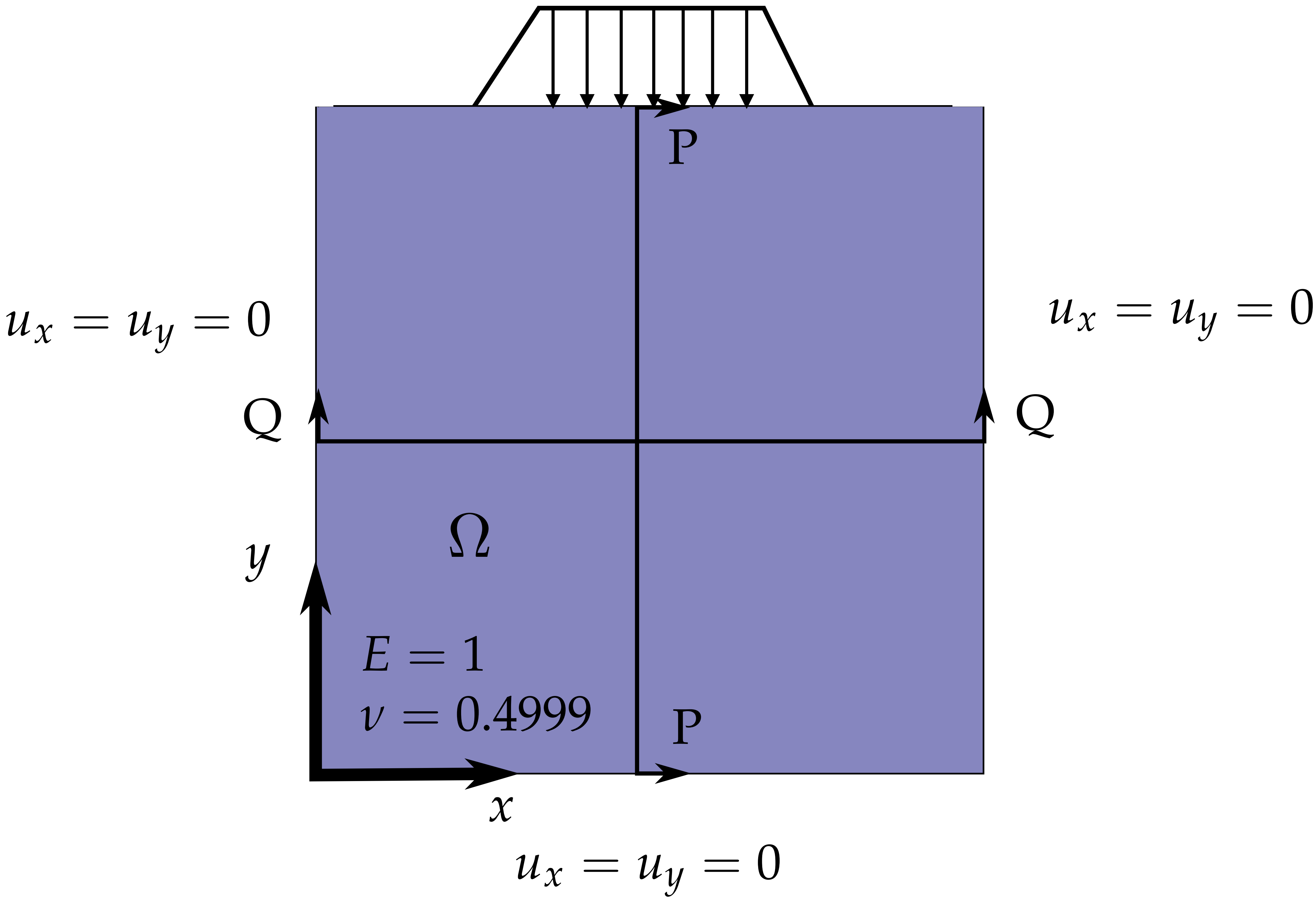
$$\text{MaxEnt } u_x$$

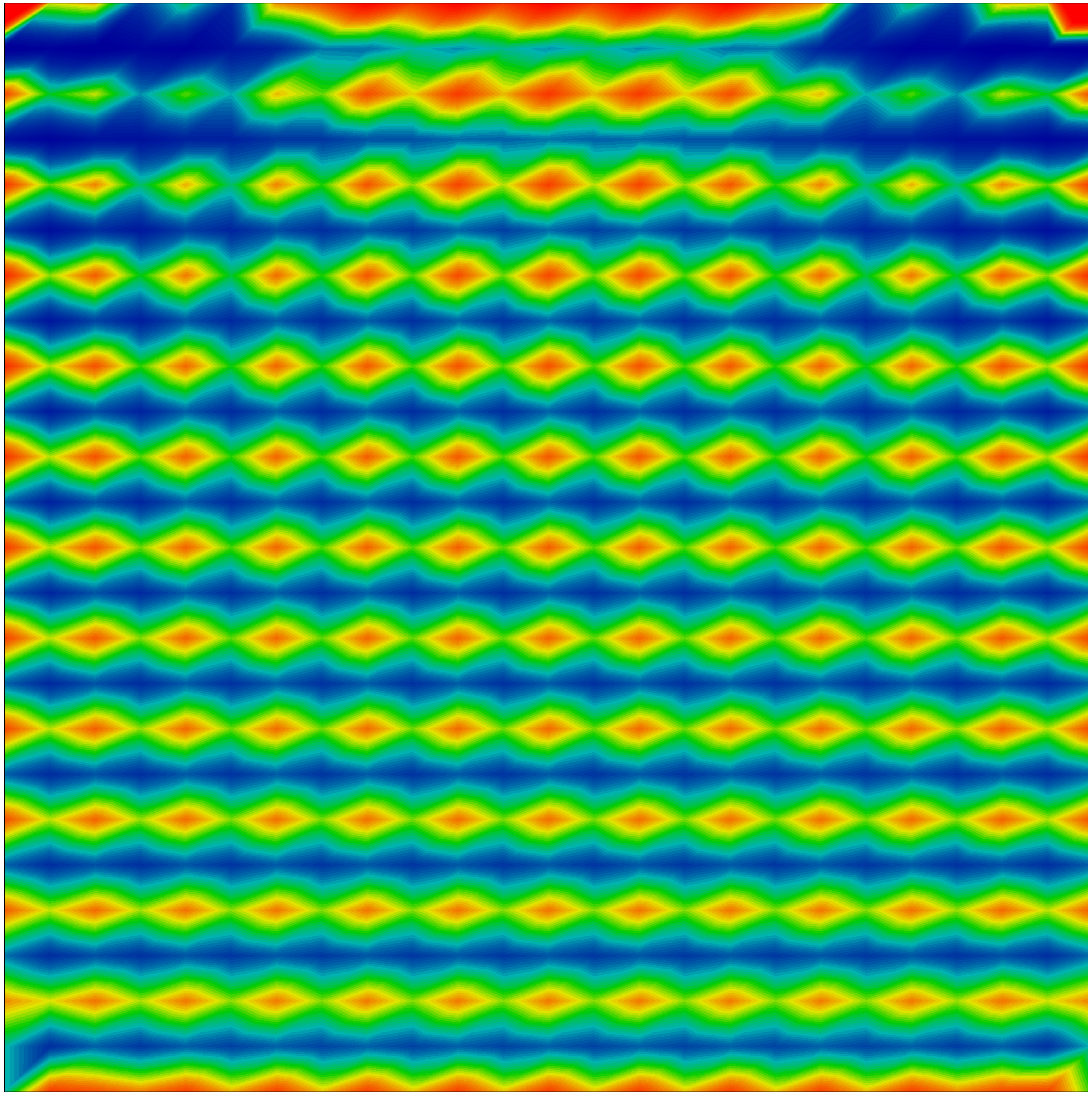




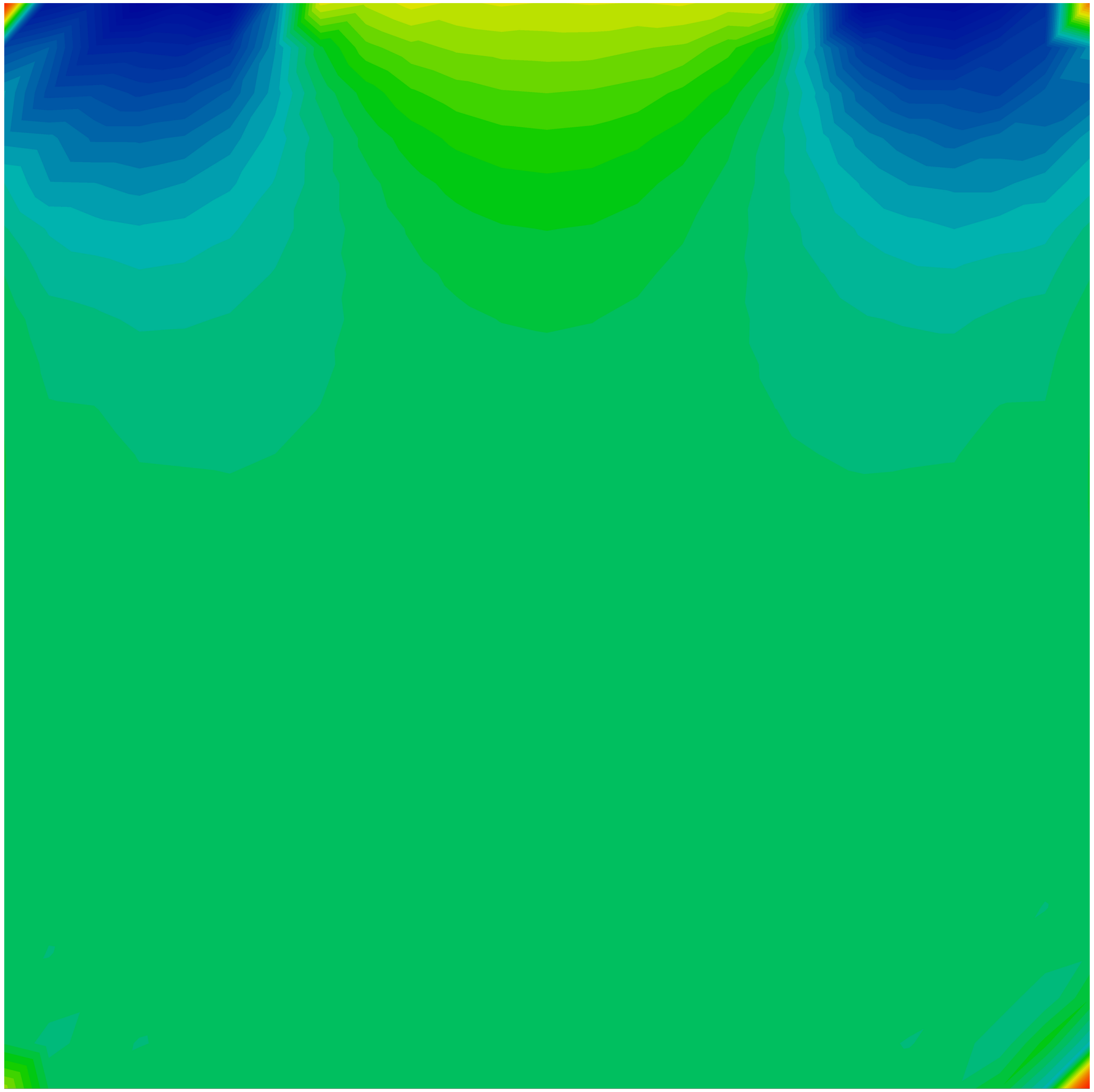
MINI u_x
u
1
0.75
0.5
0.25
0
-0.205614

Rigid Punch





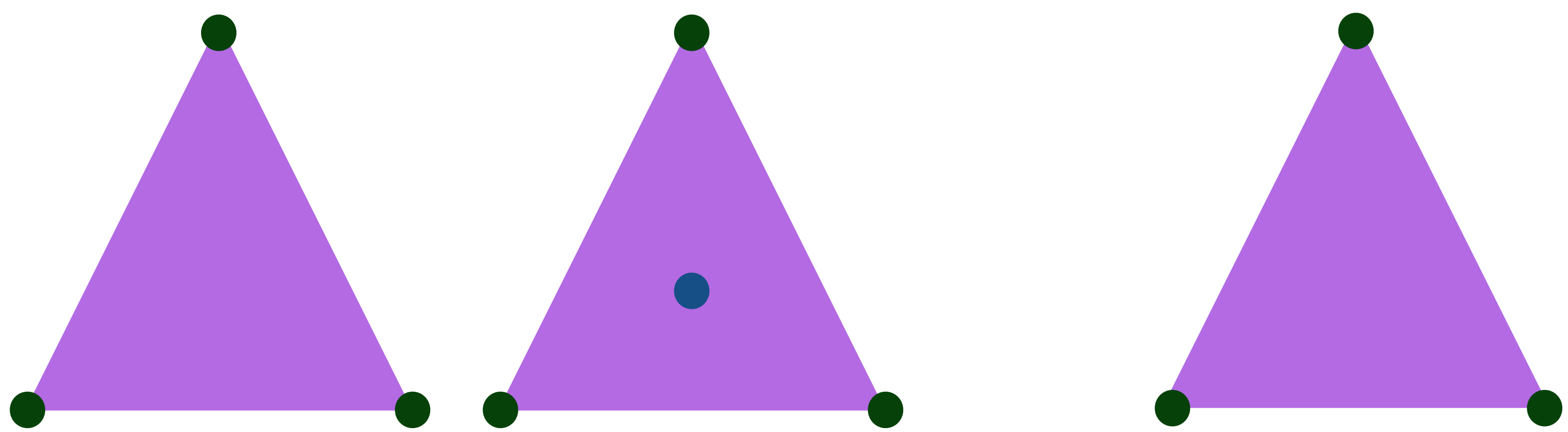
No bubbles



Bubbles

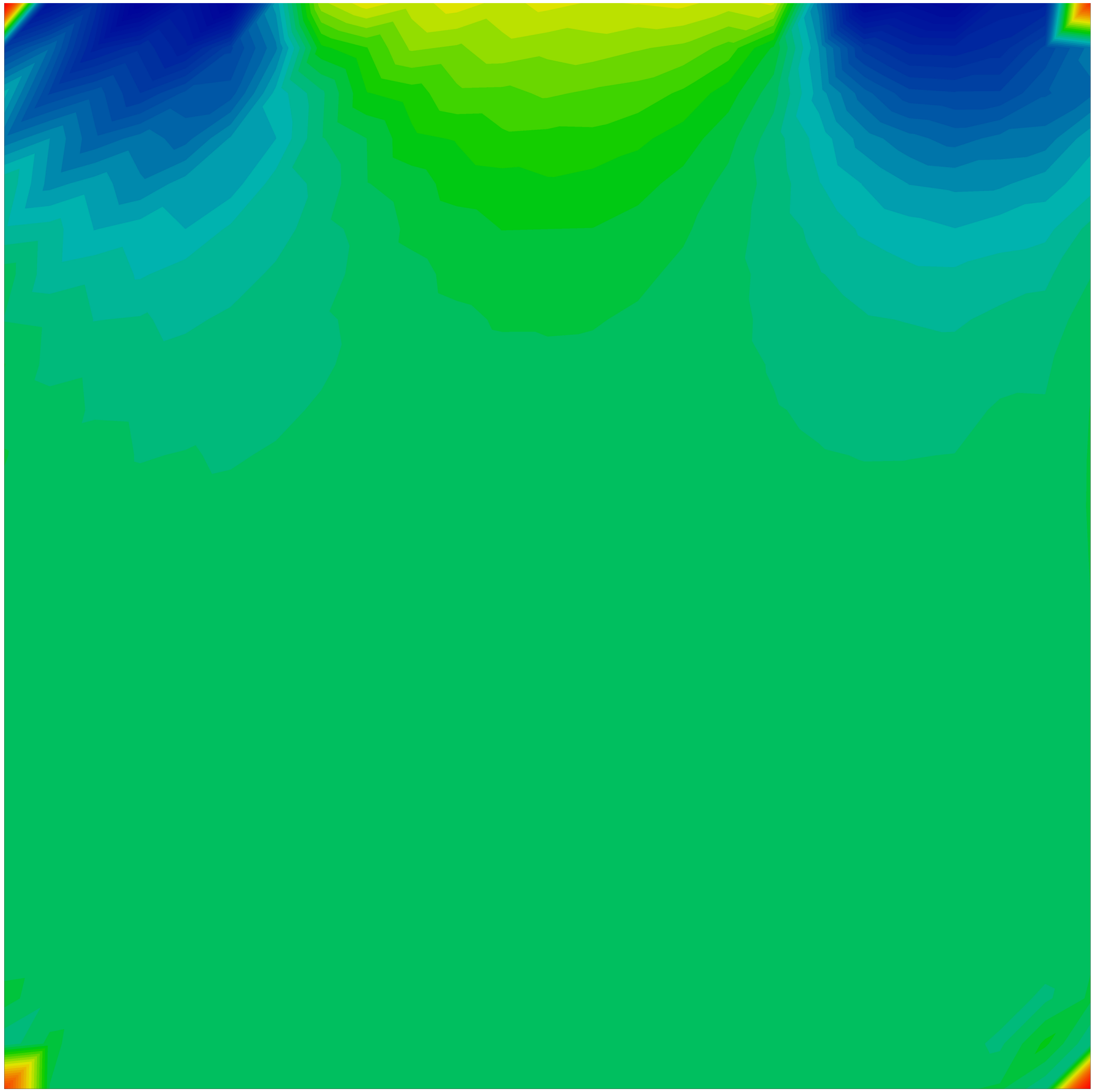
MINI* element

$[CG_1 \oplus B_3]^2$ CG_1

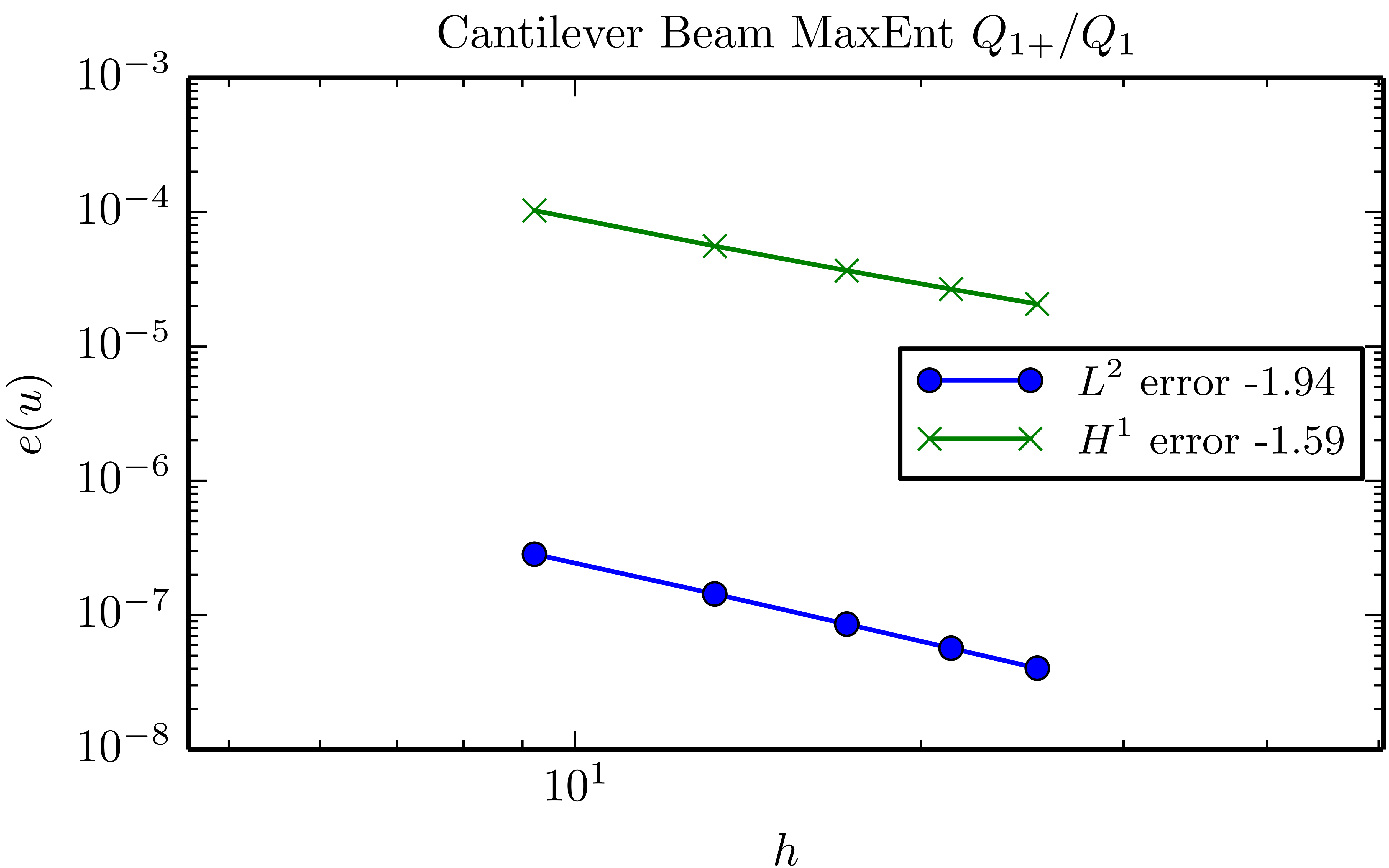


$\mathcal{U}_h$ $\mathcal{P}_h$

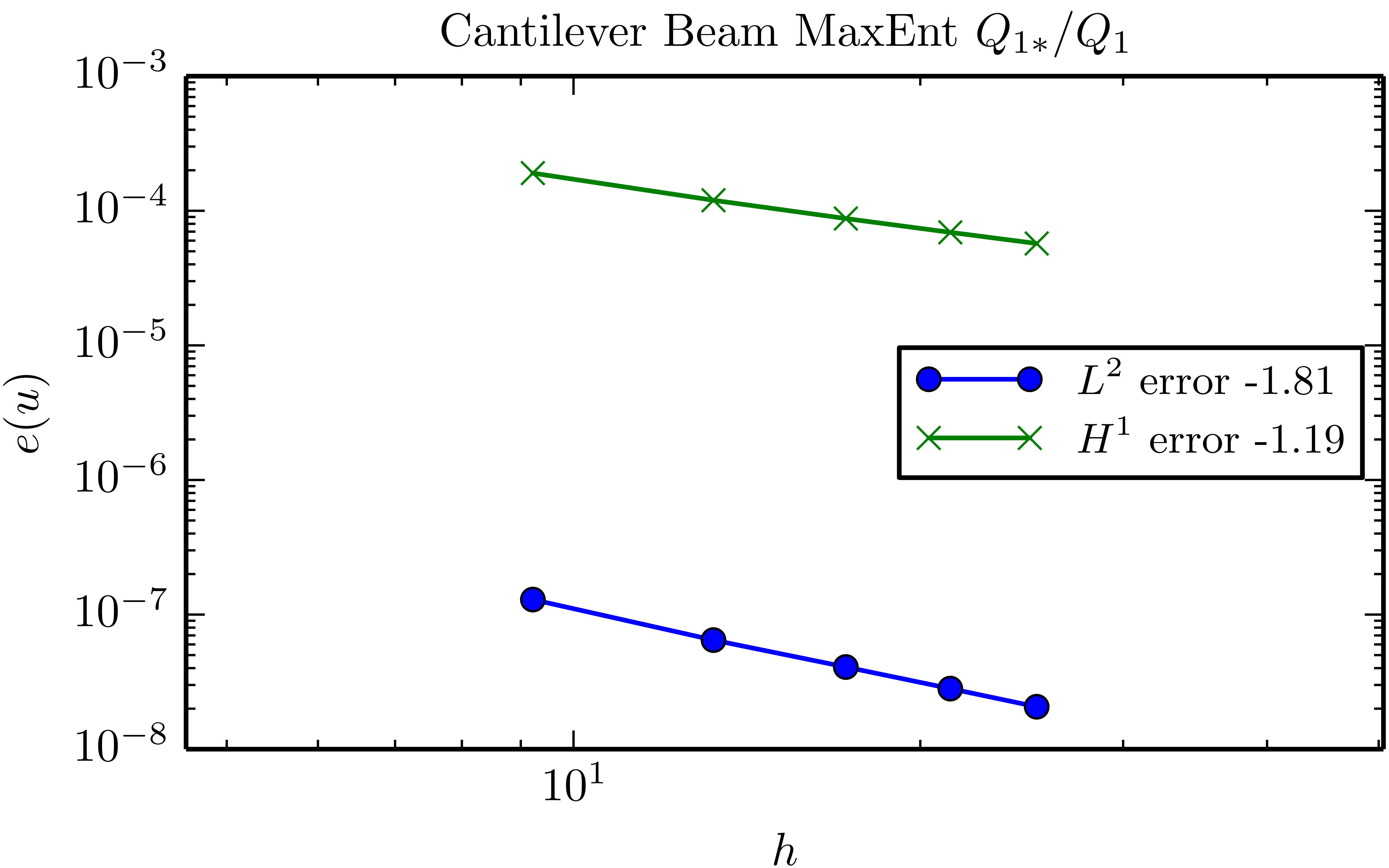
Kim and Lee, 2000, 10.1023/A:1018973303935



Half bubbles



Cantilever Beam MaxEnt Q_{1+}/Q_1
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
$e(u)$
10^1
h
L^2 error -1.94
H^1 error -1.59



Cantilever Beam MaxEnt Q_{1*}/Q_1
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
$e(u)$
10^1
h
L^2 error -1.81
H^1 error -1.19

Summary

- In an upcoming paper we show:
  - That the approach works for arbitrary basis function constructions, as long as the inf-sup condition holds.
    - Crucial to this is the use of accurate integration rules for polynomial orders greater than one.
  - Numerical inf-sup test results.
  - More convergence results.
  - Effectiveness of approach for incompressible hyperelastic problems. Greatly reduced sensitivity to distorted meshes.

Acknowledgements

Jack S. Hale would like to acknowledge the support of the Marie Curie COFUND scheme through the FNR, the structural position under the chair of Prof. Stephane P. A. Bordas at the University of Luxembourg, and the doctoral training scheme at Imperial College London/EPSRC.

Alejandro Ortiz would like to acknowledge the financial support of Fondecyt Grant No. 11110389 'Development and Assessment of An Efficient Numerical Method for Simulation of Nearly Incompressible Large Deformations Problems in Solid Mechanics'.

Jack S. Hale and Christian J. Cyron are grateful for the foreign scholar grants funded by Fondecyt Grant No. 11110389 to visit the Universidad de Chile.